The Necessary and Sufficient Condition for Some Tensors which Satisfy a Generalized $\mathcal{BP}$ – Recurrent Finsler Space

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Abstract. In this paper, we introduce and study the necessary and sufficient condition for some tensors which satisfy the generalized recurrence property in the sense of Berwald which is characterized by the following condition [1]

$$\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh}) , \quad P^i_{jkh} \neq 0 ,$$

where $\mathcal{B}_l$ is Berwald's covariant differential operator concerning x^l , λ_l and μ_l are known as recurrence vectors. We use Cartan's second kind covariant derivative and Berwald covariant derivative together to get the necessary and sufficient condition for some tensors.

Keywords. Finsler space, Curvature tensor, Generalized $\mathcal{BP}$ – recurrent space, h –covariant derivative, Berwald covariant derivative.

1.Introduction

Finsler geometry is a kind of differential geometry in pure mathematics which originated by Finsler in 1918. It is usually considered as a generalization of Riemannian geometry. Pandey et al. [19] introduced a generalized H – recurrent Finsler space, Youssef et al. [16] discussed some vector fields in Finsler Geometry.

Verma [20], Pandey and Verma [18], Sarangi and Goswami [14], Mishra and Lodhi [5], Dikshit [21], Qasem [6], Hussien [17] and Mohammed [3] introduced and discussed the necessary and sufficient condition for some tensors which satisfy a recurrence property. Qasem and Abdallah [7,8], Qasem and Saleem [9], Qasem and Baleedi [12], Awed [2], Qasem and Al– Qashbari [10,11] introduced the necessary and sufficient condition for some tensors which satisfy a generalized $\mathcal{BR}$ – recurrent space, a generalized $\mathcal{BN}$ –recurrent space, a generalized $\mathcal{BK}$ – recurrent space, a generalized P^h – recurrent space and generalized H^h– , R^h – recurrent space, respectively.

The vector y_i defined by

(1.1) $y_i = g_{ij}(x,y)y^j$

The two sets of quantities g_{ij} and its associative g^{ij} , which are components of a metric tensor connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k , \\ 0 & if\ j \neq k . \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta^i_k\, y^k = y^i$, b) $\delta^i_j g_{ir} = g_{jr}$

and c) $\delta^i_k y_i = y_k$

The vector y_i also satisfies the following relations

(1.4) a) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$ and

b) $\dot\partial_i y^j = \delta^j_i$

The tensor C_{ijk} is defined by

(1.5) $C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij}$

which called *(h)hv-torsion tensor* [15] and its associative $C^{\ i}_{jk}$ is symmetric in its lower indices and called *(v)hv- torsion tensor*, these tensors satisfy the following:

(1.6) a) $C_{ijk} = g_{hj}C^h_{ik}$ and b) $C^i_{ri} = C_r$

É. Cartan *h- covariant differentiation* (*Cartan*'s *second kind covariant differentiation*) with respect to x^k is given by [13]

(1.7) $X^i_{|k} := \partial_k X^i - (\dot\partial_r X^i)G^{\ r}_k + X^r\Gamma^{*\ i}_{rk}$.

the metric tensor g_{ij} is covariant constant with respect to h –covariant derivative, *i.e.*

(1.8) $g_{ij|k} = 0$

The h – covariant derivative of the vector y^i vanish identically $i.e.$

(1.9) $y^i{}_{|k} = 0$.

Let us consider new connection parameter G^i_{jk} of L. Berwald which are connected with Cartan's connection parameter Γ^{*i}_{jk} by

(1.10) $G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|h}\, y^h$.

Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [13]

(1.11) $\mathcal{B}_k T^i_j := \partial_k T^i_j - (\dot\partial_r T^i_j)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$.

The commutation formula of Berwald's covariant differentiation with respect to x^h and the partial differentiation with respect to y^k for an arbitrary tensor T^i_j is given by

(1.12) $(\dot\partial_k \mathcal{B}_h - \mathcal{B}_h \dot\partial_k)T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

(1.13) $\mathcal{B}_k y^i = 0$

But in general, Berwald covariant derivative of the metric tensor g_{ij} non-vanishing and given by

(1.14) $\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_h C_{ijk}$.

The tensor P^i_{jkh} called $hv-$ *curvature tensor* (*Cartan's second curvature tensor*) defined by

(1.15) $P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr}P^r_{kh} - C^i_{jh|k}$

Satisfies the relation

(1.16) $P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh}y^j = P^i_{kh} = C^i_{kh|r}y^r$,

Where P^i_{kh} is $(v)hv-$ *torsion tensor*

(1.17) $P^i_{jk}y^j = 0$

using (1.16) in (1.10), we get

(1.18) $P^i_{kh} = G^i_{kh} - \Gamma^{*i}_{kh}$.

The $P-$ Ricci tensor P_{jk} is given by

(1.19) $P^i_{jki} = P_{jk}$

The associate tensor P_{ijkh} of the $hv-$curvature tensor P^i_{jkh} is given by [13]

(1.20) $P_{ijkh} = g_{ir}\, P^r_{jkh}$

The tensor $(P_{ij} - P_{ji})$ is given by

(1.21) $P_{ijkh}\, g^{kh} = P_{ij} - P_{ji}$

The curvature vector P_k is given by

(1.22) $P^i_{ki} = P_k$

The curvature scalar P is given by

(1.23) $P_k\, y^k = P$

the $hv-$ curvature tensor P^i_{jkh} satisfies the following:

(1.24) $P^i_{jkh} - P^i_{kjh} = -S^i_{jkh|r}\, y^r$

and

(1.25) $P^i_{jkh} - P^i_{kjh} = C^i_{kh|j} + C^i_{sj}P^s_{kh} - j/k$.

The tensor R^i_{jkh} called $h-$*curvature tensor (Cartan's third curvature tensor)* defined by [13]

(1.26) $R^i_{jkh} := \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_\ell \Gamma^{*i}_{jh})G^\ell_k + G^i_{jm}(\partial_h G^m_k - G^{\ m}_{h\ell}G^\ell_k) + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - h/k.$

Satisfy the relations

(1.27) $R^i_{jkh} = K^i_{jkh} + C^i_{jm}\, H^m_{kh}$

and

(1.28) $R^i_{jkh}\, y^j = H^i_{kh}$

The associate tensor R_{ijkh} of the $h-$curvature tensor R^i_{jkh} is satisfies

(1.29) a) $R_{ijkh} = g_{ir}R^r_{jkh}$ and

b) $R_{ijkh} = K_{ijkh} + C_{ijm}\, H^m_{kh}$

$R-$Ricci tensor R_{jk}, and the curvature vector R_k are given by

(1.30) $R_{jk} = K_{jk} + C^r_{jm}\, H^m_{kr}$

and

(1.31) $R_{jk}\, y^j = R_k$.

The tensor S^i_{jkh} is called $v-$ *curvature tensor (Cartan's first curvature tensor)* defined by [13]

(1.32) $S^i_{jkh} = C^i_{rk}C^r_{jh} - C^i_{rh}C^r_{jk}$.

The associate curvature tensor S_{ijkh} is given by

(1.33) $S_{ijkh} = g_{ri}\, S^r_{jkh}$.

2.A Generalized $\mathcal{B}P-$ Recurrent Space

Let us consider a Finsler space F_n which Cartan's second curvature tensor P^i_{jkh} satisfies the condition [1]

(2.1) $\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$, $P^i_{jkh} \neq 0$

where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m, λ_m and μ_m are non$-$zero covariant vectors field, this space introduced by Alaa el al. [1], they called it a generalized $\mathcal{B}P-$recurrent space and denoted it briefly by $G(\mathcal{B}P)-RF_n$.

Let us consider a $G(\mathcal{B}P)-RF_n$ which is characterized by the condition (2.1)

Transvecting the condition (2.1) by g_{im} , using (1.20), (1.14) and (1.3b), we get

(2.2) $\mathcal{B}_l P_{mjkh} = \lambda_l P_{mjkh} + \mu_l(g_{jm}g_{kh} - g_{km}g_{jh}) + 2P^i_{jkh}y^t\mathcal{B}_t C_{iml}$

Transvecting the condition (2.1) by y^j, using (1.16), (1.13), (1.3a) and (1.1), we get

(2.3) $\mathcal{B}_l P^i_{kh} = \lambda_l P^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h)$.

Contracting the indices i and h in the condition (2.1), using (1.19) and (1.3b), we get

(2.4) $\mathcal{B}_l P_{jk} = \lambda_l P_{jk}$.

Contracting the indices i and h in the condition (2.3), using (1.22), (1.1) and (1.3c), we get

(2.5) $\mathcal{B}_l P_k = \lambda_l P_k$.

3. Necessary And Sufficient Condition

We obtained the necessary and sufficient condition for some tensors which satisfies a generalized recurrent in a $(\mathcal{B}P)-RF_n$.

For a Riemannian space V_4 , the projective curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined as follows [4]

(3.1) $P^i_{jkh} = R^i_{jkh} - \frac{1}{3}(\delta^i_h R_{jk} - \delta^i_k R_{jh})$.

Transvecting the condition (3.1) by g_{im} , using (1.20), (1.29a) and (1.3b), we get

(3.2) $P_{mjkh} = R_{mjkh} - \frac{1}{3}(R_{jk}g_{mh} - R_{jh}g_{mk})$.

Taking the covariant derivative for (3.1) with respect to x^l in the sense of Berwald and using the condition (2.1), we get

(3.3) $\mathcal{B}_l R^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh}) + \frac{1}{3}\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh})$.

Using the condition (3.1) in (3.3), we get

(3.4) $\mathcal{B}_l R^i_{jkh} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - \frac{1}{3}\lambda_l(\delta^i_h R_{jk} - \delta^i_k R_{jh}) + \frac{1}{3}\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh})$.

This shows that

$$\mathcal{B}_l R^i_{jkh} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$

If and only if

$$\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh}) = \lambda_l(\delta^i_h R_{jk} - \delta^i_k R_{jh}).$$

Thus, we conclude

Theorem 3.1. *In* $G(\mathcal{B}P)-RF_n$, *for* $n = 4$, *Cartan's third curvature tensor* R^i_{jkh} *is a generalized recurrent if and only if the tensor* $(\delta^i_h R_{jk} - \delta^i_k R_{jh})$ *is recurrent.*

Taking the covariant derivative for (3.2) with respect to x^l in the sense of Berwald and using the condition (2.2), we get

(3.5) $\mathcal{B}_l R_{mjkh} = \lambda_l P_{mjkh} + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$
$+\mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$
$+\frac{1}{3}\mathcal{B}_l(R_{jk}g_{mh} - R_{jh}g_{mk})$.

Using the condition (3.2) in (3.5), we get

(3.6) $\mathcal{B}_l R_{mjkh} = \lambda_l R_{mjkh} + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$.
$+\mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$
$+\frac{1}{3}\mathcal{B}_l(R_{jk}g_{mh} - R_{jh}g_{mk})$
$-\frac{1}{3}\lambda_l(R_{jk}g_{mh} - R_{jh}g_{mk})$

This shows that

(3.7) $\mathcal{B}_l R_{mjkh} = \lambda_l R_{mjkh}$
$+\mu_l(g_{jm}g_{kh} - g_{km}g_{jh}) + P^i_{jkh} y^t \mathcal{B}_t C_{iml}$

If and only if

$\mathcal{B}_l(R_{jk}g_{mh} - R_{jh}g_{mk}) = \lambda_l(R_{jk}g_{mh} - R_{jh}g_{mk})$.

Thus, we conclude

Theorem 3.2. *In $G(\mathcal{B}P) - RF_n$, for $n = 4$, the associate curvature tensor R_{mjkh} of Cartan's third curvature tensor is given by (3.7) if and only if the tensor $(R_{jk}g_{mh} - R_{jh}g_{mk})$ is recurrent.*

Transvecting the condition (3.1) by y^j, using (1.16), (1.28) and (1.31), we get

(3.8) $P^i_{kh} = H^i_{kh} - \frac{1}{3}(\delta^i_h R_k - \delta^i_k R_h)$.

Taking the covariant derivative for (3.8) with respect to x^l in the sense of Berwald and using the condition (2.3), we get

(3.9) $\mathcal{B}_l H^i_{kh} = \lambda_l P^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h)$
$+\frac{1}{3}\mathcal{B}_l(\delta^i_h R_k - \delta^i_k R_h)$.

Using (3.8) in (3.9), we get

(3.10) $\mathcal{B}_l H^i_{kh} = \lambda_l H^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h)$
$+\frac{1}{3}\mathcal{B}_l(\delta^i_h R_k - \delta^i_k R_h)$
$-\frac{1}{3}\lambda_l(\delta^i_h R_k - \delta^i_k R_h)$.

This shows that

(3.11) $\mathcal{B}_l H^i_{kh} = \lambda_l H^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h)$

If and only if

$\mathcal{B}_l(\delta^i_h R_k - \delta^i_k R_h) = \lambda_l(\delta^i_h R_k - \delta^i_k R_h)$.

Thus, we conclude

Theorem 3.3. *In $G(\mathcal{B}P) - RF_n$, for $n = 4$, the torsion tensor H^i_{kh} of Berwald is given by (3.11) if and only if the tensor $(\delta^i_h R_k - \delta^i_k R_h)$ is recurrent.*

Using (1.27) and (1.30) in (3.1), we get

(3.12) $P^i_{jkh} = K^i_{jkh} + C^i_{jm} H^m_{kh}$
$-\frac{1}{3}\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\}$.

Taking the covariant derivative for (3.12) with respect to x^l in the sense of Berwald and using the condition (2.1), we get

(3.13) $\mathcal{B}_l K^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$
$+\frac{1}{3}\mathcal{B}_l\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\}$
$-\mathcal{B}_l(C^i_{jm} H^m_{kh})$

Using (3.12) in (3.13), we get

(3.14) $\mathcal{B}_l K^i_{jkh} = \lambda_l K^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$
$-\frac{1}{3}\lambda_l\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\}$
$+\frac{1}{3}\mathcal{B}_l\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\}$
$+\lambda_l(C^i_{jm} H^m_{kh}) - \mathcal{B}_l(C^i_{jm} H^m_{kh})$

This shows that

$\mathcal{B}_l K^i_{jkh} = \lambda_l K^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$

If and only if

(3.15) $\frac{1}{3}\mathcal{B}_l\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\} - \mathcal{B}_l(C^i_{jm} H^m_{kh})$
$= \frac{1}{3}\lambda_l\{(K_{jk} + C^r_{jm} H^m_{kr})\delta^i_h - (K_{jh} + C^r_{jm} H^m_{hr})\delta^i_k\} - \lambda_l(C^i_{jm} H^m_{kh})$.

Thus, we conclude

Theorem 3.4. *In $G(\mathcal{B}P) - RF_n$, for $n = 4$, Cartan's fourth curvature tensor K^i_{jkh} is a generalized recurrent if and only if Eq. (3.15) holds .*

Using (1.29b) and (1.30) in (3.2), we get

(3.16) $P_{mjkh} = K_{mjkh} + C_{mjs} H^s_{kh}$
$-\frac{1}{3}\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh}$
$-(K_{jh} + C^r_{js} H^s_{hr})g_{mk}\}$

Taking the covariant derivative for (3.16) with respect to x^l in the sense of Berwaldand using the condition (2.2), we get

(3.17) $\mathcal{B}_l K_{mjkh} = \lambda_l P_{mjkh} - \mathcal{B}_l(C_{mjs} H^s_{kh})$
$+\mu_l(g_{jm}g_{kh} - g_{km}g_{jh}) + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$
$+\frac{1}{3}\lambda_l\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh} - (K_{jh} + C^r_{js} H^s_{hr})g_{mk}\}$

Using (3.16) in (3.17), we get

(3.18) $\mathcal{B}_l K_{mjkh} = \lambda_l K_{mjkh}$
$+\mu_l(g_{jm}g_{kh} - g_{km}g_{jh}) + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$
$+\lambda_l(C_{mjs} H^s_{kh}) - \mathcal{B}_l(C_{mjs} H^s_{kh}) -$
$\frac{1}{3}\lambda_l\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh} - (K_{jh} + C^r_{js} H^s_{hr})g_{mk}\} +$
$\frac{1}{3}\mathcal{B}_l\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh} - (K_{jh} + C^r_{js} H^s_{hr})g_{mk}\}$

This shows that

$\mathcal{B}_l K_{mjkh} = \lambda_l K_{mjkh} + \mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$

If and only if

(3.19) $\frac{1}{3}\mathcal{B}_l\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh}$
$-(K_{jh} + C^r_{js} H^s_{hr})g_{mk}\} - \mathcal{B}_l(C_{mjs} H^s_{kh})$
$= \frac{1}{3}\lambda_l\{(K_{jk} + C^r_{js} H^s_{kr})g_{mh}$
$-(K_{jh} + C^r_{js} H^s_{hr})g_{mk}\} - \lambda_l(C_{mjs} H^s_{kh})$
$-2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$.

Thus, we conclude

Theorem 3.5. *In $G(\mathcal{B}P) - RF_n$, for $n = 4$, the associate curvature tensor K_{mjkh} is a generalized recurrent if and only if Eq. (3.19) holds .*

Transvecting (1.24) by g_{im}, using (1.20), (1.8) and (1.33), we get

(3.20) $P_{mjkh} - P_{mkjh} = (-S_{mjkh|r} y^r)$

Taking the covariant derivative for (3.20) with respect to x^l in the sense of Berwald and using the condition (2.2), we get

(3.21) $\mathcal{B}_l(-S_{mjkh|r} y^r) = \lambda_l(P_{mjkh} - P_{mkjh})$
$+2\mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$
$+2(P^i_{jkh} - P^i_{kjh}) y^t \mathcal{B}_t C_{iml}$.

Using (3.20) in (3.21), we get

$$\mathcal{B}_l(-S_{mjkh|r}y^r) = \lambda_l(-S_{mjkh|r}y^r)$$
$$-\alpha_l(g_{jm}g_{kh} - g_{km}g_{jh})$$
$$+2(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml}$$

Where $\alpha = -2\mu_l$

This shows that

$$\mathcal{B}_l(S_{mjkh|r}y^r) = \lambda_l(S_{mjkh|r}y^r)$$
$$+\alpha_l(g_{jm}g_{kh} - g_{km}g_{jh})$$

If and only if

$$(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml} = 0.$$

Thus, we conclude

Theorem 3.6. *In $G(\mathcal{B}P) - RF_n$, the tensor $(S_{mjkh|r}y^r)$ is a generalized recurrent if and only if $(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml} = 0$.*

Transvecting (1.25) by g_{im}, using (1.20), (1.8) and (1.6a), we get

(3.22) $$P_{mjkh} - P_{mkjh} = (C_{mkh|j} + P^s_{kh}C_{msj} - j/k).$$

Taking the covariant derivative for (3.24) with respect to x^l in the sense of Berwald and using the condition (2.2), we get

(3.23) $$\mathcal{B}_l(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$$
$$= \lambda_l(P_{mjkh} - P_{mkjh})$$
$$+2\mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$$
$$+2(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml}.$$

Using (3.22) in (3.23), we get

$$\mathcal{B}_l(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$$
$$= \lambda_l(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$$
$$+\alpha_l(g_{jm}g_{kh} - g_{km}g_{jh})$$
$$+2(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml}$$

Where $\alpha_l = 2\mu_l$

This shows that

$$\mathcal{B}_l(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$$
$$= \lambda_l(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$$
$$+\alpha_l(g_{jm}g_{kh} - g_{km}g_{jh})$$

If and only if

$$(P^i_{jkh} - P^i_{kjh})y^t\mathcal{B}_tC_{iml} = 0.$$

Thus, we conclude

Theorem 3.7. *In $G(\mathcal{B}P) - RF_n$, the tensor $(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$ is a generalized recurrent if and only if $(P^i_{jkh} - P^i_{kjh})\, y^t\, \mathcal{B}_t\, C_{iml} = 0$.*

Transvecting the condition (2.2) by g^{kh}, using (1.21), (1.2), (1.3b) and put $(g^{kh}g_{kh} = 1)$, we get

$$\mathcal{B}_l(P_{mj} - P_{jm}) = \lambda_l(P_{mj} - P_{jm})$$
$$+ 2g^{kh}P^i_{jkh}y^t\mathcal{B}_tC_{iml} + P_{mjkh}\,\mathcal{B}_l\, g^{kh}.$$

This shows that

(3.24) $$\mathcal{B}_l(P_{mj} - P_{jm}) = \lambda_l(P_{mj} - P_{jm})$$

If and only if

(3.25) $$2g^{kh}\, P^i_{jkh}y^t\mathcal{B}_tC_{iml} + P_{mjkh}\,\mathcal{B}_l\, g^{kh} = 0.$$

Thus, we conclude

Theorem 3.8. *In $G(\mathcal{B}P) - RF_n$, the tensor $(P_{mj} - P_{jm})$ behaves as recurrent if and only if Eq. (3.25) holds .*

Contracting the indices i and h in the condition (1.25), using (1.19) and (1.6b), we get

(3.26) $$P_{jk} - P_{kj} = C_{k|j} - C^i_{sj}P^s_{ki} - j/k.$$

Taking the covariant derivative for (3.26) with respect to x^l in the sense of Berwald and using (3.24), we get

(3.27) $$\mathcal{B}_l(C_{k|j} - C^i_{sj}P^s_{ki} - j/k) = \lambda_l(P_{mj} - P_{jm}).$$

Using (3.26) in (3.27), we get

$$\mathcal{B}_l(C_{k|j} - C^i_{sj}P^s_{ki} - j/k)$$
$$= \lambda_l(C_{k|j} - C^i_{sj}P^s_{ki} - j/k).$$

Thus, we conclude

Theorem 3.9. *In $G(\mathcal{B}P) - RF_n$, the tensor $(C_{k|j} - C^i_{sj}P^s_{ki} - j/k)$ behaves as recurrent [provided Eq. (3.25) holds].*

Differentiating (2.3) partially with respect to y^j, using (1.4b), (1.5) and (1.4a), we get

(3.28) $$\dot\partial_j(\mathcal{B}_lP^i_{kh}) = (\dot\partial_j\lambda_l)P^i_{kh} + \lambda_l(\dot\partial_jP^i_{kh})$$
$$+(\dot\partial_j\mu_l)(y^ig_{kh} - \delta^i_ky_h)$$
$$+\mu_l(\delta^i_jg_{kh} + 2y^iC_{jkh} - \delta^i_kg_{jh}).$$

Using the commutation formula exhibited by (1.12) for (H^i_{kh}) in (3.28), we get

$$\mathcal{B}_l(\dot\partial_jP^i_{kh}) + P^r_{kh}G^i_{jlr} - P^i_{rh}G^r_{jlk} - P^i_{kr}G^r_{jlh}$$
$$= (\dot\partial_j\lambda_l)P^i_{kh} + \lambda_l(\dot\partial_jP^i_{kh})$$
$$+(\dot\partial_j\mu_l)(y^ig_{kh} - \delta^i_ky_h)$$
$$+\mu_l(\delta^i_jg_{kh} - \delta^i_kg_{jh}) + 2\mu_ly^iC_{jkh}.$$

This shows that

$$\mathcal{B}_l(\dot\partial_jP^i_{kh}) = \lambda_l(\dot\partial_jP^i_{kh}) + \mu_l(\delta^i_jg_{kh} - \delta^i_kg_{jh})$$

if and only if

(3.29) $$P^r_{kh}G^i_{jlr} - P^i_{rh}G^r_{jlk} - P^i_{kr}G^r_{jlh} - (\dot\partial_j\lambda_l)P^i_{kh}$$
$$-(\dot\partial_j\mu_l)(y^ig_{kh} - \delta^i_ky_h) - 2\mu_ly^iC_{jkh} = 0.$$

Thus, we conclude

Theorem 3.10. *In $G(\mathcal{B}P) - RF_n$, the tensor $(\dot\partial_jP^i_{kh})$ is a generalized recurrent if and only if Eq.(3.28) holds.*

Differentiating (2.4) partially with respect to y^h, we get

(3.30) $$\dot\partial_h(\mathcal{B}_lP_{jk}) = (\dot\partial_h\lambda_l)P_{jk} + \lambda_l(\dot\partial_hP_{jk}).$$

Using the commutation formula exhibited by (1.12) for (P_{jk}) in (3.30), we get

$$\mathcal{B}_l(\dot\partial_hP_{jk}) + P_{rk}G^r_{hlj} - P_{jr}G^r_{hlk}$$
$$= (\dot\partial_h\lambda_l)P_{jk} + \lambda_l(\dot\partial_hP_{jk}).$$

This shows that

(3.31) $$\mathcal{B}_l(\dot\partial_hP_{jk}) = \lambda_l(\dot\partial_hP_{jk})$$

if and only if

(3.32) $$P_{rk}G^r_{hlj} - P_{jr}G^r_{hlk} - (\dot\partial_h\lambda_l)P_{jk} = 0.$$

Differentiating (2.5) partially with respect to y^j, we get

(3.33) $$\dot\partial_j(\mathcal{B}_lP_k) = (\dot\partial_j\lambda_l)P_k + \lambda_l(\dot\partial_jP_k).$$

Using the commutation formula exhibited by (1.12) for (P_k) in (3.33), we get

$$\mathcal{B}_l(\dot\partial_jP_k) - P_rG^r_{jlk} = (\dot\partial_j\lambda_l)P_k + \lambda_l(\dot\partial_jP_k)$$

This shows that

(3.34) $$\mathcal{B}_l(\dot\partial_jP_k) = \lambda_l(\dot\partial_jP_k)$$

if and only if

(3.35) $$P_rG^r_{jlk} + (\dot\partial_j\lambda_l)P_k = 0\,.$$

The equations (3.31) and (3.34) show that the tensors $(\dot\partial_hP_{jk})$ and $(\dot\partial_jP_k)$ behave as recurrent if and only if (3.32) and (3.35), respectively holds.

Thus, we conclude

Theorem 3.11. *In $G(\mathcal{B}P)-RF_n$, the tensors $(\dot{\partial}_h P_{jk})$ and $(\dot{\partial}_j P_k)$ behave as recurrent if and only if Eqs.(3.32) - (3.35), respectively holds.*

Using the definition of the covariant derivative in the sense of Berwald for the hv −curvature tensor P^i_{jkh} exhibited by (1.11) with respect to x^l, we get

$$(3.36)\quad \mathcal{B}_l P^i_{jkh} = \partial_l P^i_{jkh} + P^r_{jkh}G^i_{rl} - P^i_{rkh}G^r_{jl} - P^i_{jrh}G^r_{kl} - P^i_{jkr}G^r_{hl} - (\dot{\partial}_r P^i_{jkh})G^r_l .$$

Using (1.18) in (3.36), we get

$$\mathcal{B}_l P^i_{jkh} = \partial_l P^i_{jkh} + P^r_{jkh}\Gamma^{*i}_{rl} - P^i_{rkh}\Gamma^{*r}_{jl} - P^i_{jrh}\Gamma^{*r}_{kl} - P^i_{jrh}\Gamma^{*r}_{hl} - (\dot{\partial}_r P^i_{jkh})G^r_l + P^r_{jkh}P^i_{rl} - P^i_{rkh}P^r_{jl} - P^i_{jrh}P^r_{kl} - P^i_{jkr}P^r_{hl}.$$

Using the definition of the $h-$covariant derivative in the sense of Cartan exhibited by (1.7) with respect to x^l in above equation, we get

$$(3.37)\quad \mathcal{B}_l P^i_{jkh} = P^i_{jkh|l} + P^r_{jkh}P^i_{rl} - P^i_{rkh}P^r_{jl} - P^i_{jrh}P^r_{kl} - P^i_{jkr}P^r_{hl}$$

Using the condition (2.1) in (3.37), we get

$$P^i_{jkh|l} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - P^r_{jkh}P^i_{rl} + P^i_{rkh}P^r_{jl} + P^i_{jrh}P^r_{kl} + P^i_{jkr}P^r_{hl} .$$

This shows that

$$P^i_{jkh|l} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$

if and only if

$$(3.38)\quad P^r_{jkh}P^i_{rl} - P^i_{rkh}P^r_{jl} - P^i_{jrh}P^r_{kl} - P^i_{jkr}P^r_{hl} = 0.$$

Thus, we conclude

Theorem 3.12. *In $G(\mathcal{B}P)-RF_n$, the hv −curvature tensor P^i_{jkh} is a generalized recurrent in the sense of Cartan if and only if Eq.(3.38) holds .*

Using the definition of the covariant derivative in the sense of Berwald for the associative curvature tensor P_{mjkh} exhibited by (1.11) with respect to x^l, we get

$$(3.39)\quad \mathcal{B}_l P_{mjkh} = \partial_l P_{mjkh} + P_{rjkh}G^r_{ml} - P_{mrkh}G^r_{jl} - P_{mjrh}G^r_{kl} - P_{mjkr}G^r_{hl} - (\dot{\partial}_r P_{mjkh})G^r_l .$$

Using (1.18) in (3.39), we get

$$\mathcal{B}_l P_{mjkh} = \partial_l P_{mjkh} + P_{rjkh}\Gamma^{*r}_{ml} - P_{mrkh}\Gamma^{*r}_{jl} - P_{mjrh}\Gamma^{*r}_{kl} - P_{mjkr}\Gamma^{*r}_{hl} - (\dot{\partial}_r P_{mjkh})G^r_l + P_{rjkh}P^r_{ml} - P_{mrkh}P^r_{jl} - P_{mjrh}P^r_{kl} - P_{mjkr}P^r_{hl} .$$

Using the definition of the $h-$covariant derivative in the sense of Cartan exhibited by (1.7) with respect to x^l in above equation, we get

$$(3.40)\quad \mathcal{B}_l P_{mjkh} = P_{mjkh|l} + P_{rjkh}P^r_{ml} - P_{mrkh}P^r_{jl} - P_{mjrh}P^r_{kl} - P_{mjkr}P^r_{hl} .$$

Using the condition (2.2) in (3.40), we get

$$P_{mjkh|l} = \lambda_l P_{mjkh} + \mu_l(g_{jm}g_{kh} - g_{km}g_{jh}) + 2P^i_{jkh}y^t\mathcal{B}_t C_{iml} - P_{rjkh}P^r_{ml} + P_{mrkh}P^r_{jl} + P_{mjrh}P^r_{kl} + P_{mjkr}P^r_{hl} .$$

This shows that

$$P_{mjkh|l} = \lambda_l P_{mjkh} + \mu_l(g_{jm}g_{kh} - g_{km}g_{jh})$$

if and only if

$$(3.41)\quad P_{rjkh}P^r_{ml} - P_{mrkh}P^r_{jl} - P_{mjrh}P^r_{kl} - P_{mjkr}P^r_{hl} + 2P^i_{jkh}y^t\mathcal{B}_t C_{iml} = 0 .$$

Thus, we conclude

Theorem 3.13. *In $G(\mathcal{B}P)-RF_n$, the associative curvature tensor P_{mjkh} is a generalized recurrent in the sense of Cartan if and only if (3.40) holds .*

Transvecting (3.37) by y^j, using (1.16), (1.13), (1.9) and (1.17), we get

$$(3.42)\quad \mathcal{B}_l P^i_{kh} = P^i_{kh|l} + P^r_{kh}P^i_{rl} - P^i_{rh}P^r_{kl} - P^i_{kr}P^r_{hl}$$

Using (2.3) in (3.42), we get

$$P^i_{kh|l} = \lambda_l P^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h) - P^r_{kh}P^i_{rl} + P^i_{rh}P^r_{kl} + P^i_{kr}P^r_{hl} .$$

This shows that

$$(3.43)\quad P^i_{kh|l} = \lambda_l P^i_{kh} + \mu_l(y^i g_{kh} - \delta^i_k y_h)$$

if and only if

$$(3.44)\quad P^r_{kh}P^i_{rl} - P^i_{rh}P^r_{kl} - P^i_{kr}P^r_{hl} = 0.$$

Thus, we conclude

Theorem 3.14. *In $G(\mathcal{B}P)-RF_n$, the (v)hv – torsion tensor P^i_{kh} is given by (3.43) in the sense of Cartan if and only if Eq. (3.44) holds .*

Contracting the indices i and h in (3.37) and using (1.19), we get

$$(3.45)\quad \mathcal{B}_l P_{jk} = P_{jk|l} + P^r_{jki}P^i_{rl} - P_{rk}P^r_{jl} - P_{jr}P^r_{kl} - P^i_{jkr}P^r_{il} .$$

Using (2.4) in (3.45), we get

$$P_{jk|l} = \lambda_l P_{jk} - P^r_{jki}P^i_{rl} + P_{rk}P^r_{jl} + P_{jr}P^r_{kl} + P^i_{jkr}P^r_{il} .$$

This shows that

$$(3.46)\quad P_{jk|l} = \lambda_l P_{jk}$$

if and only if

$$(3.47)\quad P^r_{jki}P^i_{rl} - P^i_{jkr}P^r_{il} - P_{rk}P^r_{jl} - P_{jr}P^r_{kl} = 0 .$$

Contracting the indices i and h in (3.42) and using (1.22), we get

$$(3.48)\quad \mathcal{B}_l P_k = P_{k|l} + P^r_{ki}P^i_{rl} - P_r P^r_{kl} - P^i_{kr}P^r_{il} .$$

Using (2.5) in (3.48), we get

$$P_{k|l} = \lambda_l P_k - P^r_{ki}P^i_{rl} + P_r P^r_{kl} + P^i_{kr}P^r_{il} .$$

This shows that

$$(3.49)\quad P_{k|l} = \lambda_l P_k$$

if and only if

$$(3.50)\quad P^r_{ki}P^i_{rl} - P^i_{kr}P^r_{il} - P_r P^r_{kl} = 0 .$$

The equations (3.46) and (3.49) show that the P – Ricci tensor P_{jk}, the curvature vector P_k behave as recurrent in since of Cartan if and only if (3.47) and (3.50), respectively holds.

Thus, we conclude

Theorem 3.15. *In $G(\mathcal{B}P)-RF_n$, P – Ricci tensor P_{jk}, the curvature vector P_k behave as recurrent in the since of Cartan if and only if Eqs.(3.47)-(3.50), respectively holds.*

4. Conclusion

We obtained the necessary and sufficient condition for Cartan's third curvature tensor R^i_{jkh}, Cartan's fourth curvature tensor K^i_{jkh}, the associate curvature tensor K_{mjkh} , the tensor $(S_{mjkh|r}y^r)$, the tensor $(C_{mkh|j} + P^s_{kh}C_{msj} - j/k)$ and the tensor $(\dot{\partial}_j P^i_{kh})$ which are generalized recurrent. We obtained the necessary and sufficient condition for the some tensors $(P_{mj} - P_{jm})$, $(C_{k|j} - C^i_{sj}P^s_{ki} - j/k)$, $(\dot{\partial}_h P_{jk})$, $(\dot{\partial}_j P_k)$, P – Ricci tensor P_{jk} and the curvature vector P_k which behave as recurrent in $G(\mathcal{B}P)-$

RF_n. We obtained the necessary and sufficient condition for some tensors by using Cartan's second kind covariant derivative with Berwald covariant derivative.

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